On the Super-Renormalizablity of Gauge Models in the Causal Approach

D. R. Grigore, [1]

Department of Theoretical Physics
Institute for Physics and Nuclear Engineering "Horia Hulubei"
Bucharest-Măgurele, P. O. Box MG 6, ROMÂNIA

Abstract

We consider some typical gauge models in the causal approach: Yang-Mills and pure massless gravity up to the second order of the perturbation theory. We prove that the loop contributions are coboundaries, up to super-renormalizable terms in the Yang-Mills case; this means that the ultra-violet behavior is better than expected from power counting considerations. For the pure massless gravity we prove that the loop contributions are coboundaries so the model is essentially classical. We conjecture that such a result should be true in higher orders of the perturbation theory also. This result should make easier the problem of constructive quantum field theory.

[1] e-mail: grigore@theory.nipne.ro

1 Introduction

The general framework of perturbation theory consists in the construction of the chronological products: for every set of Wick monomials $W_1(x_1), \dots, W_n(x_n)$ acting in some Fock space $\mathcal{H}$ one associates the operator $T^{W_1,\dots,W_n}(x_1,\dots,x_n)$; all these expressions are in fact distribution-valued operators called chronological products. It will be convenient to use another notation: $T(W_1(x_1),\dots,W_n(x_n))$. These operators are constrained by Bogoliubov axioms [1], [4], [2]; we prefer the setting from [2]. (An equivalent point of view uses retarded products [15].) The construction of the chronological products can be done recursively according to Epstein-Glaser prescription [4], [5] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora prescription [11] (which reduces the renormalization procedure to the process of extension of distributions). These products are not uniquely defined but there are some natural limitation on the arbitrariness. If this arbitrariness does not grow with the order n of the perturbation theory then we say that the theory is renormalizable; the most popular point of view is that only such theories are physically meaningful.

Gauge theories describe particles of higher spin. Usually such theories are not renormalizable. However, one can save renormalizability using ghost fields. Such theories are defined in a Fock space $\mathcal{H}$ with indefinite metric, generated by physical and un-physical fields (called *ghost fields*). One selects the physical states assuming the existence of an operator Q called *gauge charge* which verifies $Q^2 = 0$ and such that the *physical Hilbert space* is by definition $\mathcal{H}_{\rm phys} \equiv Ker(Q)/Im(Q)$. The space $\mathcal{H}$ is endowed with a grading (usually called *ghost number*) and by construction the gauge charge is raising the ghost number of a state. Moreover, the space of Wick monomials in $\mathcal{H}$ is also endowed with a grading which follows by assigning a ghost number to every one of the free fields generating $\mathcal{H}$. The graded commutator d_Q of the gauge charge with any operator A of fixed ghost number

$$d_Q A = [Q, A] \tag{1.1}$$

is raising the ghost number by a unit. Because

$$d_Q^2 = 0 \tag{1.2}$$

it means that d_Q is a co-chain operator in the space of Wick polynomials. From now on $[\cdot,\cdot]$ denotes the graded commutator.

A gauge theory assumes also that there exists a Wick polynomial of null ghost number $T(x)$ called *the interaction Lagrangian* such that

$$[Q, T] = i\partial_\mu T^\mu \tag{1.3}$$

for some other Wick polynomials T^μ. This relation means that the expression T leaves invariant the physical states, at least in the adiabatic limit. Indeed, if this is true we have:

$$T(f)\ \mathcal{H}_{\rm phys} \subset \ \mathcal{H}_{\rm phys} \tag{1.4}$$

up to terms which can be made as small as desired (making the test function f flatter and flatter). In all known models one finds out that there exist a chain of Wick polynomials $T^{\mu},\ T^{\mu\nu},\ T^{\mu\nu\rho},\ldots$ such that:

$$[Q,T] = i\partial_{\mu}T^{\mu}, \quad [Q,T^{\mu}] = i\partial_{\nu}T^{\mu\nu}, \quad [Q,T^{\mu\nu}] = i\partial_{\rho}T^{\mu\nu\rho},\ldots \tag{1.5}$$

It so happens that for all these models the expressions $T^{\mu\nu},\ T^{\mu\nu\rho},\ldots$ are completely antisymmetric in all indexes; it follows that the chain of relation stops at the step 4 (if we work in four dimensions). We can also use a compact notation T^{I} where I is a collection of indexes $I = [\nu_1,\ldots,\nu_p]$ $(p = 0,1,\ldots,)$ and the brackets emphasize the complete antisymmetry in these indexes. All these polynomials have the same canonical dimension

$$\omega(T^{I}) = \omega_0,\ \forall I \tag{1.6}$$

and because the ghost number of $T \equiv T^{\emptyset}$ is supposed null, then we also have:

$$gh(T^{I}) = |I|. \tag{1.7}$$

One can write compactly the relations (1.5) as follows:

$$d_{Q}T^{I} = i\ \partial_{\mu}T^{I\mu}. \tag{1.8}$$

For concrete models the equations (1.5) can stop earlier: for instance in the Yang-Mills case we have $T^{\mu\nu\rho} = 0$ and in the case of gravity $T^{\mu\nu\rho\sigma} = 0$. If the interaction Lagrangian T is Lorentz invariant, then one can prove that the expressions $T^{I},\ |I| > 0$ can be taken Lorentz covariant.

Now we can construct the chronological products

$$T^{I_1,\ldots,I_n}(x_1,\ldots,x_n) \equiv T(T^{I_1}(x_1),\ldots,T^{I_n}(x_n))$$

according to the recursive procedure. We say that the theory is gauge invariant in all orders of the perturbation theory if the following set of identities generalizing (1.8):

$$d_{Q}T^{I_1,\ldots,I_n} = i\sum_{l=1}^{n}(-1)^{s_l}\frac{\partial}{\partial x_l^{\mu}}T^{I_1,\ldots,I_l\mu,\ldots,I_n} \tag{1.9}$$

are true for all $n \in \mathbb{N}$ and all $I_1,\ldots,I_n$. Here we have defined

$$s_l \equiv \sum_{j=1}^{l-1}|I|_j. \tag{1.10}$$

We introduce some cohomology terminology. We consider a *cochains* to be an ensemble of distribution-valued operators of the form $C^{I_1,\ldots,I_n}(x_1,\ldots,x_n),\ n = 1,2,\cdots$ (usually we impose some supplementary symmetry properties) and define the derivative operator δ according to

$$(\delta C)^{I_1,\ldots,I_n} = \sum_{l=1}^{n}(-1)^{s_l}\frac{\partial}{\partial x_l^{\mu}}C^{I_1,\ldots,I_l\mu,\ldots,I_n}. \tag{1.11}$$

We can prove that

$$\delta^2 = 0. \tag{1.12}$$

Next we define

$$s = d_Q - i\delta, \qquad \bar{s} = d_Q + i\delta \tag{1.13}$$

and note that

$$s\bar{s} = \bar{s}s = 0. \tag{1.14}$$

We call *relative cocycles* the expressions C verifying

$$sC = 0 \tag{1.15}$$

and a *relative coboundary* an expression C of the form

$$C = \bar{s}B. \tag{1.16}$$

The relation (1.9) is simply the cocycle condition

$$sT = 0. \tag{1.17}$$

The purpose of this paper is to investigate if this condition implies that, at least some contributions of T, are in fact coboundaries. Coboundaries are trivial from the physical point of view: if we consider two physical states Ψ, Ψ' then

$$< \Psi, \bar{s}B\Psi' >=< Q\Psi, B\Psi' > - < \Psi, BQ\Psi' > +i < \Psi, \delta B\Psi' >\to 0 \tag{1.18}$$

(in the adiabatic limit).

We will consider here only the second order of the perturbation theory and prove that for Yang-Mills models the loop contributions are coboundaries, up to super-renormalizable terms (i.e. terms with a better ultra-violet behavior than given by power counting); for massless gravity the situation is even better, i.e. the loop contributions are strictly a coboundary i.e. the theory is essentially classical. This follows from the fact that in the loop expansion the 0-loop (or tree) contribution corresponds to the classical theory [3].

In the next Section we present the description of the free fields use, mainly to fix the notations. In Section 3 we give Bogoliubov axioms for the second order of the perturbation theory; in Subsection 3.3 we give the basic distributions with causal support appearing for loop contributions in the second order of the perturbation theory. In Sections 4 and 5 we prove the cohomology result for Yang-Mills and gravity.

2 Free Fields

We summarize some results from [9].

2.1 Massless Particles of Spin 1 (Photons)

We consider a vector space $\mathcal{H}$ of Fock type generated (in the sense of Borchers theorem) by the vector field v_μ (with Bose statistics) and the scalar fields $u, \tilde{u}$ (with Fermi statistics). The Fermi fields are usually called *ghost fields*. We suppose that all these (quantum) fields are of null mass. Let Ω be the vacuum state in $\mathcal{H}$. In this vector space we can define a sesquilinear form $< \cdot, \cdot >$ in the following way: the (non-zero) 2-point functions are by definition:

$$
\begin{aligned}
&< \Omega, v_\mu(x_1) v_\mu(x_2) \Omega >= i\ \eta_{\mu\nu}\ D_0^{(+)}(x_1 - x_2), \\
< \Omega, u(x_1) \tilde{u}(x_2) \Omega >= -i\ D_0^{(+)}(x_1 - x_2) \qquad &< \Omega, \tilde{u}(x_1) u(x_2) \Omega >= i\ D_0^{(+)}(x_1 - x_2)
\end{aligned}
\tag{2.1}
$$

and the n-point functions are generated according to Wick theorem. Here $\eta_{\mu\nu}$ is the Minkowski metrics (with diagonal $1, -1, -1, -1$) and $D_0^{(+)}$ is the positive frequency part of the Pauli-Jordan distribution D_0 of null mass. To extend the sesquilinear form to $\mathcal{H}$ we define the conjugation by

$$
v_\mu^\dagger = v_\mu, \qquad u^\dagger = u, \qquad \tilde{u}^\dagger = -\tilde{u}. \tag{2.2}
$$

Now we can define in $\mathcal{H}$ the operator Q according to the following formulas:

$$
\begin{aligned}
[Q, v_\mu] = i\ \partial_\mu u, \qquad [Q, u] = 0, \qquad &[Q, \tilde{u}] = -i\ \partial_\mu v^\mu \\
&Q\Omega = 0
\end{aligned}
\tag{2.3}
$$

where by $[\cdot, \cdot]$ we mean the graded commutator. One can prove that Q is well defined. Indeed, we have the causal commutation relations

$$
[v_\mu(x_1), v_\mu(x_2)] = i\ \eta_{\mu\nu}\ D_0(x_1 - x_2)\ \cdot I, \qquad [u(x_1), \tilde{u}(x_2)] = -i\ D_0(x_1 - x_2)\ \cdot I \tag{2.4}
$$

and the other commutators are null. The operator Q should leave invariant these relations, in particular

$$
[Q, [v_\mu(x_1), \tilde{u}(x_2)]] + \text{cyclic permutations} = 0 \tag{2.5}
$$

which is true according to (2.3). It is useful to introduce a grading in $\mathcal{H}$ as follows: every state which is generated by an even (odd) number of ghost fields and an arbitrary number of vector fields is even (resp. odd). We denote by $|f|$ the ghost number of the state f. We notice that the operator Q raises the ghost number of a state (of fixed ghost number) by an unit. The usefulness of this construction follows from:

Theorem 2.1 *The operator Q verifies $Q^2 = 0$. The factor space $Ker(Q)/Ran(Q)$ is isomorphic to the Fock space of particles of zero mass and helicity 1 (photons).*

2.2 Massive Particles of Spin 1 (Heavy Bosons)

We repeat the whole argument for the case of massive photons i.e. particles of spin 1 and positive mass.

We consider a vector space $\mathcal{H}$ of Fock type generated (in the sense of Borchers theorem) by the vector field v_μ, the scalar field Φ (with Bose statistics) and the scalar fields $u, \tilde{u}$ (with Fermi statistics). We suppose that all these (quantum) fields are of mass $m > 0$. In this vector space we can define a sesquilinear form $< \cdot, \cdot >$ in the following way: the (non-zero) 2-point functions are by definition:

$$
\begin{gathered}
< \Omega, v_\mu(x_1) v_\mu(x_2) \Omega >= i\ \eta_{\mu\nu}\ D_m^{(+)}(x_1 - x_2), \quad < \Omega, \Phi(x_1) \Phi(x_2) \Omega >= -i\ D_m^{(+)}(x_1 - x_2) \\
< \Omega, u(x_1) \tilde{u}(x_2) \Omega >= -i\ D_m^{(+)}(x_1 - x_2), \qquad < \Omega, \tilde{u}(x_1) u(x_2) \Omega >= i\ D_m^{(+)}(x_1 - x_2)
\end{gathered}
\tag{2.6}
$$

and the n-point functions are generated according to Wick theorem. Here $D_m^{(+)}$ is the positive frequency part of the Pauli-Jordan distribution D_m of mass m. To extend the sesquilinear form to $\mathcal{H}$ we define the conjugation by

$$
v_\mu^\dagger = v_\mu, \qquad u^\dagger = u, \qquad \tilde{u}^\dagger = -\tilde{u}, \qquad \Phi^\dagger = \Phi. \tag{2.7}
$$

Now we can define in $\mathcal{H}$ the operator Q according to the following formulas:

$$
\begin{gathered}
[Q, v_\mu] = i\ \partial_\mu u, \qquad [Q, u] = 0, \qquad [Q, \tilde{u}] = -i\ (\partial_\mu v^\mu + m\ \Phi) \qquad [Q, \Phi] = i\ m\ u, \\
Q\Omega = 0.
\end{gathered}
\tag{2.8}
$$

One can prove that Q is well defined. We have a result similar to the first theorem of this Section:

Theorem 2.2 *The operator Q verifies $Q^2 = 0$. The factor space $Ker(Q)/Ran(Q)$ is isomorphic to the Fock space of particles of mass m and spin 1 (massive photons).*

2.3 The Generic Yang-Mills Case

The situations described above (of massless and massive photons) are susceptible of the following generalizations. We can consider a system of r_1 species of particles of null mass and helicity 1 if we use in the first part of this Section r_1 triplets $(v_a^\mu, u_a, \tilde{u}_a), a \in I_1$ of massless fields; here I_1 is a set of indices of cardinal r_1. All the relations have to be modified by appending an index a to all these fields.

In the massive case we have to consider r_2 quadruples $(v_a^\mu, u_a, \tilde{u}_a, \Phi_a), a \in I_2$ of fields of mass m_a; here I_2 is a set of indexes of cardinal r_2.

We can consider now the most general case involving fields of spin not greater that 1. We take $I = I_1 \cup I_2 \cup I_3$ a set of indexes and for any index we take a quadruple $(v_a^\mu, u_a, \tilde{u}_a, \Phi_a), a \in I$ of fields with the following conventions: (a) For $a \in I_1$ we impose $\Phi_a = 0$ and we take the masses to be null $m_a = 0$; (b) For $a \in I_2$ we take the all the masses strictly positive: $m_a > 0$; (c) For $a \in I_3$ we take $v_a^\mu, u_a, \tilde{u}_a$ to be null and the fields $\Phi_a \equiv \phi_a^H$ of mass $m_a^H \geq 0$. The fields ϕ_a^H are called *Higgs fields*.

If we define $m_a = 0, \forall a \in I_3$ then we can define in $\mathcal{H}$ the operator Q according to the following formulas for all indexes $a \in I$:

$$\begin{aligned} [Q, v_a^\mu] = i\ \partial^\mu u_a, \qquad [Q, u_a] = 0, \\ [Q, \tilde{u}_a] = -i\ (\partial_\mu v_a^\mu + m_a\ \Phi_a) \qquad [Q, \Phi_a] = i\ m_a\ u_a, \\ Q\Omega = 0. \end{aligned} \tag{2.9}$$

If we consider matter fields also i.e some set of Dirac fields with Fermi statistics: $\Psi_A, A \in I_4$ then we impose

$$d_Q \Psi_A = 0. \tag{2.10}$$

2.4 Massless Particles of Spin 2 (Gravitons)

We consider the vector space $\mathcal{H}$ of Fock type generated (in the sense of Borchers theorem) by the symmetric tensor field $h_{\mu\nu}$ (with Bose statistics) and the vector fields $u^{\rho}, \tilde{u}^{\sigma}$ (with Fermi statistics). We suppose that all these (quantum) fields are of null mass. In this vector space we can define a sesquilinear form $< \cdot, \cdot >$ in the following way: the (non-zero) 2-point functions are by definition:

$$
\begin{aligned}
< \Omega, h_{\mu\nu}(x_1) h_{\rho\sigma}(x_2)\Omega >= -\frac{i}{2}\ (\eta_{\mu\rho}\ \eta_{\nu\sigma} + \eta_{\nu\rho}\ \eta_{\mu\sigma} - \eta_{\mu\nu}\ \eta_{\rho\sigma})\ D_0^{(+)}(x_1 - x_2), \\
< \Omega, u_{\mu}(x_1)\tilde{u}_{\nu}(x_2)\Omega >= i\ \eta_{\mu\nu}\ D_0^{(+)}(x_1 - x_2), \\
< \Omega, \tilde{u}_{\mu}(x_1) u_{\nu}(x_2)\Omega >= -i\ \eta_{\mu\nu}\ D_0^{(+)}(x_1 - x_2)
\end{aligned}
\tag{2.11}
$$

and the n-point functions are generated according to Wick theorem. Here $\eta_{\mu\nu}$ is the Minkowski metrics (with diagonal $1, -1, -1, -1$) and $D_0^{(+)}$ is the positive frequency part of the Pauli-Jordan distribution D_0 of null mass. To extend the sesquilinear form to $\mathcal{H}$ we define the conjugation by

$$
h_{\mu\nu}^{\dagger} = h_{\mu\nu}, \qquad u_{\rho}^{\dagger} = u_{\rho}, \qquad \tilde{u}_{\sigma}^{\dagger} = -\tilde{u}_{\sigma}. \tag{2.12}
$$

Now we can define in $\mathcal{H}$ the operator Q according to the following formulas:

$$
\begin{aligned}
[Q, h_{\mu\nu}] = -\frac{i}{2}\ (\partial_{\mu} u_{\nu} + \partial_{\nu} u_{\mu} - \eta_{\mu\nu}\partial_{\rho} u^{\rho}), \qquad [Q, u_{\mu}] = 0, \qquad [Q, \tilde{u}_{\mu}] = i\ \partial^{\nu} h_{\mu\nu} \\
Q\Omega = 0
\end{aligned}
\tag{2.13}
$$

where by $[\cdot, \cdot]$ we mean the graded commutator. One can prove that Q is well defined. Indeed, we have the causal commutation relations

$$
\begin{aligned}
[h_{\mu\nu}(x_1), h_{\rho\sigma}(x_2)] = -\frac{i}{2}\ (\eta_{\mu\rho}\ \eta_{\nu\sigma} + \eta_{\nu\rho}\ \eta_{\mu\sigma} - \eta_{\mu\nu}\ \eta_{\rho\sigma})\ D_0(x_1 - x_2)\ \cdot I, \\
[u(x_1), \tilde{u}(x_2)] = i\ \eta_{\mu\nu}\ D_0(x_1 - x_2)\ \cdot I
\end{aligned}
\tag{2.14}
$$

and the other commutators are null. The operator Q should leave invariant these relations, in particular

$$
[Q, [h_{\mu\nu}(x_1), \tilde{u}_{\sigma}(x_2)]] + \text{cyclic permutations} = 0 \tag{2.15}
$$

which is true according to (2.3). Then we have:

Theorem 2.3 *The operator Q verifies $Q^2 = 0$. The factor space $Ker(Q)/Im(Q)$ is isomorphic to the Fock space of particles of zero mass and helicity 2 (gravitons).*

3 General Gauge Theories

We give here the essential ingredients of perturbation theory for $n = 2$. First we consider that the canonical dimension of the vector and scalar fields $v_a^\mu, u_a, \tilde{u}_a, \Phi_a$ and $h_{\mu\nu}, u_\mu, \tilde{u}_\mu$ is equal to 1 and the canonical dimension of the Dirac fields is $3/2$. A derivative applied to a field raises the canonical dimension by 1. The ghost number of the ghost fields is 1 and for the rest of the fields is null. The Fermi parity of a Fermi (Bose) field is 1 (resp. 0). The canonical dimension of a Wick monomial is additive with respect to the factors and the same is true for the ghost number and the Fermi parity.

3.1 Bogoliubov Axioms

Suppose that the Wick monomials A, B are self-adjoint: $A^\dagger = A,\ B^\dagger = B$ and of fixed Fermi parity $|A|, |B|$ and canonical dimension $\omega(A), \omega(B)$. We will consider two case: for Yang-Mills fields we can take $\omega(A), \omega(B) \leq 4$ but for gravity we have $\omega(A), \omega(B) \leq 5$. The chronological products $T(A(x), B(y))$ are verifying the following set of axioms:

- Skew-symmetry:
$$T(B(y), A(x)) = (-1)^{|A||B|} T(A(x), B(y)) \tag{3.1}$$

- Poincaré invariance: we have a natural action of the Poincaré group in the space of Wick monomials and we impose that for all elements g of the universal covering group $inSL(2, \mathbb{C})$ of the Poincaré group:
$$U_g T(A(x), B(y)) U_g^{-1} = T(g \cdot A(g \cdot x), g \cdot B(g \cdot y)) \tag{3.2}$$
where $x \mapsto g \cdot x$ is the action of $inSL(2, \mathbb{C})$ on the Minkowski space. Sometimes we can supplement this axiom with other symmetry properties, as for instance, parity invariance.

- Causality: if $x \geq y$ i.e. $(x - y)^2 \geq 0,\ x^0 - y^0 \geq 0$ then we have:
$$T(A(x), B(y)) = A(x)\ B(y); \tag{3.3}$$

- Unitarity: If we define the *anti-chronological products* according to
$$\bar{T}(A(x), B(y)) \equiv A(x)B(y) + B(y)A(x) - T(A(x), B(y)) \tag{3.4}$$
then the unitarity axiom is:
$$\bar{T}(A(x), B(y) = T(A(x), B(y))^\dagger. \tag{3.5}$$

It can be proved that this system of axioms can be supplemented with

$$T(A(x), B(y)) = \sum \quad < \Omega, T(A_1(x), B_1(y))\Omega > \quad : A_2(x)B_2(y) : \tag{3.6}$$

where $A = A_1 A_2, B = B_1 B_2$ is an arbitrary decomposition of A and resp. B in Wick submonomials and we have supposed for simplicity that no Fermi fields are present; if Fermi fields are present, then some apropriate signs do appear. This is called the *Wick expansion property*.

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products:

$$\omega(<\Omega, T(A(x), B(y))\Omega >) \leq \omega(A) + \omega(B) - 4 \tag{3.7}$$

where by $\omega(d)$ we mean the order of singularity of the (numerical) distribution d and by $\omega(W)$ we mean the canonical dimension of the Wick monomial W.

The contributions verifying

$$\omega(<\Omega, T(A(x), B(y))\Omega >) < \omega(A) + \omega(B) - 4 \tag{3.8}$$

will be called *super-renormalizable*.

The operator-valued distributions D, A, R, T admit a decomposition into loop contributions $D = \sum_l D_{(l)}$, etc. Indeed every contribution is associated with a certain Feynman graph and the integer l counts the number of the loops. Alternatively, if we consider the loop decomposition of the advanced (or retarded) products we have in fact series in $\hbar$ so the contribution corresponding to $l = 0$ (the tree contribution) is the classical part and the loop contributions $l > 0$ are the quantum corrections [3].

3.2 Second Order Cohomology

We go to the second order of perturbation theory using the *causal commutator*

$$D^{A,B}(x,y) \equiv D(A(x), B(y)) = [A(x), B(y)] \tag{3.9}$$

where $A(x), B(y)$ are arbitrary Wick monomials and, as always we mean by $[\cdot,\cdot]$ the graded commutator. These type of distributions are translation invariant i.e. they depend only on $x - y$ and the support is inside the light cones:

$$supp(D) \subset V^+ \cup V^-. \tag{3.10}$$

A theorem from distribution theory guarantees that one can causally split this distribution:

$$D(A(x), B(y)) = A(A(x), B(y)) - R(A(x), B(y)). \tag{3.11}$$

where:

$$supp(A) \subset V^+ \qquad supp(R) \subset V^-. \tag{3.12}$$

The expressions $A(A(x), B(y)), R(A(x), B(y))$ are called *advanced* resp. *retarded* products. They are not uniquely defined: one can modify them with *quasi-local terms* i.e. terms proportional with $\delta(x - y)$ and derivatives of it.

There are some limitations on these redefinitions coming from Lorentz invariance, and *power counting*: this means that we should not make the various distributions appearing in the advanced and retarded products too singular.

Then we define the *chronological product* by:

$$T(A(x), B(y)) = A(A(x), B(y)) + B(y)A(x) = R(A(x), B(y)) + A(x)B(y). \tag{3.13}$$

We consider that A and B are of the type T^I such that we have first-order gauge invariance:

$$sT = 0 \tag{3.14}$$

which is a cocycle equation. Then we define that causal commutator

$$D^{IJ}(x, y) \equiv [T^I(x), T^J(y)]; \tag{3.15}$$

we have the symmetry property

$$D^{JI}(y, x) = -(-1)^{|I||J|}\; D^{IJ}(x, y) \tag{3.16}$$

and the limitations

$$gh(D^{IJ}) = |I| + |J| \tag{3.17}$$

and power counting limitations coming from (3.7). This will be our cochain space. But $D^{IJ}(x, y)$ it is also a cocycle:

$$sD = 0 \qquad \Leftrightarrow \qquad d_Q D^{IJ} = i\frac{\partial}{\partial x^\rho} D^{I\rho,J} + i(-1)^{|I|}\frac{\partial}{\partial y^\rho} D^{I,J\rho}. \tag{3.18}$$

as it follows from (3.14). Now the key problem of gauge theories is to prove that the causal splitting of this commutator can be done such that the gauge invariance property is preserved i.e. $A^{IJ}(x, y), R^{IJ}(x, y), T^{IJ}(x, y)$ are also cocycles. In lower orders of perturbation theory this can be done elementary.

We address in this paper another question, namely if these objects are in some sense coboundaries. We will prove that this is true only for the loop contributions and in the Yang-Mills case only up to super-renormalizable terms.

3.3 Second Order Causal Distributions

We remind the fact that the Pauli-Villars distribution is defined by

$$D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x) \tag{3.19}$$

where

$$D_m^{(\pm)}(x) \sim \int dp e^{ip\cdot x}\theta(\pm p_0)\delta(p^2 - m^2) \tag{3.20}$$

such that

$$D^{(-)}(x) = -D^{(+)}(-x). \tag{3.21}$$

This distribution has causal support. In fact, it can be causally split (uniquely) into an advanced and a retarded part:

$$D = D^{\mathrm{adv}} - D^{\mathrm{ret}} \tag{3.22}$$

and then we can define the Feynman propagator and antipropagator

$$D^F = D^{\mathrm{ret}} + D^{(+)}, \qquad \bar{D}^F = D^{(+)} - D^{\mathrm{adv}}. \tag{3.23}$$

All these distributions have singularity order $\omega(D) = -2$.

For one-loop contributions in the second order we need the basic distribution

$$d_2(x) \equiv \frac{1}{2}[D_m^{(+)}(x)^2 - D_m^{(+)}(-x)^2] \tag{3.24}$$

which also with causal support and it can be causally split as above in

$$d_2 = d_2^{\mathrm{adv}} - d_2^{\mathrm{ret}} \tag{3.25}$$

and the corresponding Feynman propagators can be defined. These distributions have the singularity order $\omega(D) = 0$.

We will now consider for simplicity the case $m = 0$.

In the explicit computations some associated distributions with causal support do appear. In the Yang-Mills case we can have two derivatives distributed in two ways on the two factors $D_0^{(+)}$:

$$\begin{aligned} d_{\mu\nu}(x) &= D_0^{(+)}(x)\partial_\mu\partial_\nu D_0^{(+)}(x) - D_0^{(+)}(-x)\partial_\mu\partial_\nu D_0^{(+)}(-x) \\ f_{\mu\nu}(x) &= \partial_\mu D_0^{(+)}(x)\partial_\nu D_0^{(+)}(x) - \partial_\mu D_0^{(+)}(-x)\partial_\nu D_0^{(+)}(-x). \end{aligned} \tag{3.26}$$

It is not hard to prove that we have

$$\begin{aligned} d_{\mu\nu} &= \frac{2}{3}\left(\partial_\mu\partial_\nu - \frac{1}{4}\square\right) d_2 \\ f_{\mu\nu} &= \frac{1}{3}\left(\partial_\mu\partial_\nu + \frac{1}{2}\square\right) d_2. \end{aligned} \tag{3.27}$$

In the case of gravity we have 4 derivatives distributed on the two factors $D_0^{(+)}$. This can be done in three different ways and we obtain after some computations expressions of the type $P(\partial)d_2$ where P are polynomials of degree 4 in the derivatives. As we will see, the explicit expressions are not needed. For two-loop contributions in the second order we need

$$d_3(x) \equiv \frac{1}{6}[D_m^{(+)}(x)^3 - D_m^{(+)}(-x)^3] \tag{3.28}$$

and the associated distributions

$$\begin{aligned} d_3^{(1)}(x) &= \partial_\mu D_0^{(+)}(x)\partial_\nu D_0^{(+)}(x)\partial^\mu\partial^\nu D_0^{(+)}(x) - \partial_\mu D_0^{(+)}(-x)\partial_\nu D_0^{(+)}(-x)\partial^\mu\partial^\nu D_0^{(+)}(-x) \\ d_3^{(2)}(x) &= D_0^{(+)}(x)\partial_\mu\partial_\nu D_0^{(+)}(x)\partial^\mu\partial^\nu D_0^{(+)}(x) - D_0^{(+)}(-x)\partial_\mu\partial_\nu D_0^{(+)}(-x)\partial^\mu\partial^\nu D_0^{(+)}(-x). \end{aligned} \tag{3.29}$$

It can be proved that we have

$$d_3^{(1)}(x) = \frac{1}{4}\,\square^2 d_3(x), \qquad d_3^{(2)}(x) = \frac{1}{2}\,\square^2 d_3(x). \tag{3.30}$$

4 Yang-Mills Case

In this section we prove that the one-loop contributions appearing in the Yang-Mills case are in fact super-renormalizable i.e. the cochains are in fact coboundaries, up to super-renormalizable terms. We first need the explicit expressions for the cochains.

4.1 The Yang-Mills Lagrangian

Now we consider the framework and notations from Subsection 2.3. Then we have the following result which describes the most general form of the Yang-Mills interaction [6], [7], [8], [13]. Summation over the dummy indexes is used everywhere.

Theorem 4.1 *Let T be a relative cocycle which is tri-linear in the fields and is of canonical dimension $\omega(T) \leq 4$ and null Fermi parity. Then: (i) T is (relatively) cohomologous to a non-trivial co-cycle of the form:*

$$\begin{aligned} T = f_{abc} \left(\frac{1}{2}\ v_{a\mu}\ v_{b\nu}\ F_c^{\nu\mu} + u_a\ v_b^\mu\ \partial_\mu \tilde{u}_c \right) \\ + f'_{abc} (\Phi_a\ \phi_b^\mu\ v_{c\mu} + m_b\ \Phi_a\ \tilde{u}_b\ u_c) \\ + \frac{1}{3!}\ f''_{abc}\ \Phi_a\ \Phi_b\ \Phi_c + j_a^\mu\ v_{a\mu} + j_a\ \Phi_a; \end{aligned} \tag{4.1}$$

where we can take the constants $f_{abc} = 0$ if one of the indexes is in I_3; also $f'_{abc} = 0$ if $c \in I_3$ or one of the indexes a and b are from I_1; and $j_a^\mu = 0$ if $a \in I_3$; $j_a = 0$ if $a \in I_1$. By definition

$$\phi_a^\mu \equiv \partial^\mu \Phi_a - v_a^\mu \tag{4.2}$$

Moreover we have:

(a) The constants f_{abc} are completely antisymmetric

$$f_{abc} = f_{[abc]}. \tag{4.3}$$

(b) The expressions f'_{abc} are antisymmetric in the indexes a and b:

$$f'_{abc} = -f'_{bac} \tag{4.4}$$

and are connected to f_{abc} by:

$$f_{abc}\ m_c = f'_{cab} m_a - f'_{cba} m_b. \tag{4.5}$$

(c) The (completely symmetric) expressions $f''_{abc} = f''_{\{abc\}}$ verify

$$f''_{abc}\ m_c = \begin{cases} \frac{1}{m_c}\ f'_{abc}\ (m_a^2 - m_b^2) & for \quad a,b \in I_3, c \in I_2 \\ -\frac{1}{m_c}\ f'_{abc}\ m_b^2 & for \quad a,c \in I_2, b \in I_3. \end{cases} \tag{4.6}$$

(d) the expressions j_a^μ and j_a are bilinear in the Fermi matter fields: in tensor notations;

$$j_a^\mu = \sum_\epsilon\ \overline{\psi} t_a^\epsilon \otimes \gamma^\mu \gamma_\epsilon \psi \qquad j_a = \sum_\epsilon\ \overline{\psi} s_a^\epsilon \otimes \gamma_\epsilon \psi \tag{4.7}$$

where for every $\epsilon = \pm$ we have defined the chiral projectors of the algebra of Dirac matrices $\gamma_\epsilon \equiv \frac{1}{2}\ (I + \epsilon\ \gamma_5)$ and $t^\epsilon_a,\ s^\epsilon_a$ are $|I_4| \times |I_4|$ matrices. If M is the mass matrix $M_{AB} = \delta_{AB}\ M_A$ then we must have

$$\partial_\mu j^\mu_a = m_a\ j_a \qquad \Leftrightarrow \qquad m_a\ s^\epsilon_a = i(M\ t^\epsilon_a - t^{-\epsilon}_a\ M). \tag{4.8}$$

(ii) The relation $d_Q T = i\ \partial_\mu T^\mu$ is verified by:

$$T^\mu = f_{abc}\left(u_a\ v_{b\nu}\ F^{\nu\mu}_c - \frac{1}{2}u_a\ u_b\ d^\mu \tilde{u}_c\right) + f'_{abc}\ \Phi_a\ \phi^\mu_b\ u_c + j^\mu_a\ u_a \tag{4.9}$$

(iii) The relation $d_Q T^\mu = i\ \partial_\nu T^{\mu\nu}$ is verified by:

$$T^{\mu\nu} \equiv \frac{1}{2}f_{abc}\ u_a\ u_b\ F^{\mu\nu}_c. \tag{4.10}$$

4.2 The Generic Expressions for the One-Loop Cochains

We consider the one-loop contributions $D^{IJ}_{(1)}(x,y)$ from $D^{IJ}(x,y)$ and we write for every mass m in the game

$$D_m = D_0 + (D_m - D_0) \tag{4.11}$$

In this way we split $D^{IJ}_{(1)}(x,y)$ into a contribution $D^{IJ}_{(1)0}(x,y)$ where everywhere $D_m \mapsto D_0$ and a contribution where at least one factor D_m is replaced by the difference $D_m - D_0$. Because we have

$$\omega(D_m - D_0) = -4 \tag{4.12}$$

the second contribution will be super-renormalizable. We now consider the first contribution. By direct computations we obtain

$$D^{[\mu\nu]\emptyset}_{(1)0}(x,y) = 0 \tag{4.13}$$

$$D^{[\mu][\nu]}_{(1)0}(x,y) = (\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box)d_2(x-y)\tilde{g}_{ab}u_a(x)u_b(y) \tag{4.14}$$

$$\begin{aligned} D^{[\mu]\emptyset}_{(1)0}(x,y) = {} & (\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box)d_2(x-y)\tilde{g}_{ab}u_a(x)v_{b\nu}(y) \\ & +\partial_\nu d_2(x-y)g_{ab}[F^{\mu\nu}_a(x)u_b(y) - u_a(x)F^{\mu\nu}_b(y)] \end{aligned} \tag{4.15}$$

$$D^{\emptyset[\mu]}_{(1)0}(x,y) = -D^{[\mu]\emptyset}_{(1)0}(y,x) \tag{4.16}$$

$$\begin{aligned} D^{\emptyset\emptyset}_{(1)0}(x,y) = {} & (\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box)d_2(x-y)\tilde{g}_{ab}v_{a\mu}(x)v_{b\nu}(y) \\ & +\partial_\mu d_2(x-y)g_{ab}[-F^{\mu\nu}_a(x)v_{b\nu}(y) + \partial^\mu\tilde{u}_a(x)u_b(y) + v_{a\nu}(x)F^{\mu\nu}_b(y) - u_a(x)\partial^\mu\tilde{u}_b(y)] \\ & -d_2(x-y)g_{ab}F^{\mu\nu}_a(x)F_{b\mu\nu}(y) \\ & +\partial_\mu d_2(x-y)g^{(3)}_{ab}[\Phi_a(x)\partial^\mu\Phi_b(y) - \partial^\mu\Phi_a(x)\Phi_b(y)] - 2d_2(x-y)g^{(3)}_{ab}\partial^\mu\Phi_a(x)\partial_\mu\Phi_b(y) \\ & -i\partial_\mu d_2(x-y)[\bar{\Psi}(x)A_\epsilon \otimes \gamma^\mu\gamma_\epsilon\Psi(y) - \bar{\Psi}(y)A_\epsilon \otimes \gamma^\mu\gamma_\epsilon\Psi(x)] \\ & +\Box d_2(x-y)g^{(4)}_{ab}\Phi_a(x)\Phi_b(y) \end{aligned} \tag{4.17}$$

where we have defined some bilinear combinations in the constants appearing in the interaction Lagrangian:

$$g_{ab} = f_{pqa} f_{pqb} \qquad g^{(1)}_{ab} = f'_{pqa} f'_{pqb} \qquad g^{(2)}_{ab} = \sum_\epsilon Tr(t^\epsilon_a t^\epsilon_b) \qquad g^{(3)}_{ab} = f'_{apq} f'_{bpq}$$

$$g^{(4)}_{ab} = 2 \sum_\epsilon Tr(s^\epsilon_a s^{-\epsilon}_b) \qquad \tilde{g}_{ab} \equiv g_{ab} + \frac{1}{2} g^{(1)}_{ab} + 2 g^{(2)}_{ab} \qquad A_\epsilon = \sum_a (2 t^\epsilon_a t^\epsilon_a + s^{-\epsilon}_a s^\epsilon_a). \tag{4.18}$$

4.3 The Generic Form of the Coboundaries

In this Subsection we prove the basic result for the Yang-Mills case.

Theorem 4.2 *The expression $D^{IJ}_{(1)0}(x,y)$ is a coboundary*

$$D^{IJ}_{(1)0} = (\bar{s}B)^{IJ}. \tag{4.19}$$

Proof: It is done by providing an explicit expression of the coboundary. If we define

$$B^{[\mu\nu][\rho]}(x,y) = (\eta^{\mu\rho}\partial^\nu - \eta^{\nu\rho}\partial^\mu) d_2(x-y) h^{(0)}_{ab} u_a(x) u_b(y) \tag{4.20}$$

$$\begin{aligned} B^{[\mu\nu]\emptyset}(x,y) = &(\eta^{\mu\rho}\partial^\nu - \eta^{\nu\rho}\partial^\mu) d_2(x-y) h^{(0)}_{ab} u_a(x) v_{b\rho}(y) \\ &+ d_2(x-y) h^{(1)}_{ab} F^{\mu\nu}_a(x) u_b(y) + d_2(x-y) h^{(2)}_{ab} u_a(x) F^{\mu\nu}_b(y) \end{aligned} \tag{4.21}$$

$$\begin{aligned} B^{[\mu][\nu]}(x,y) = &(\eta^{\nu\rho}\partial^\mu - \eta^{\mu\nu}\partial^\rho) d_2(x-y) h^{(0)}_{ab} v_{a\rho}(x) u_b(y) \\ &+ (\eta^{\mu\rho}\partial^\nu - \eta^{\mu\nu}\partial^\rho) d_2(x-y) h^{(0)}_{ab} u_a(x) v_{b\rho}(y) \\ &+ d_2(x-y) h^{(3)}_{ab} [F^{\mu\nu}_a(x) u_b(y) + u_a(x) F^{\mu\nu}_b(y) \end{aligned} \tag{4.22}$$

$$\begin{aligned} B^{[\mu]\emptyset}(x,y) = &(\eta^{\nu\rho}\partial^\mu - \eta^{\mu\rho}\partial^\nu) d_2(x-y) h^{(0)}_{ab} v_{a\nu}(x) v_{b\rho}(y) \\ &+ d_2(x-y) h^{(1)}_{ab} \partial^\mu \tilde{u}_a(x) u_b(y) - d_2(x-y) h^{(2)}_{ab} v_{a\nu}(x) F^{\mu\nu}_b(y) \\ &+ d_2(x-y) h^{(3)}_{ab} [F^{\mu\nu}_a(x) v_{b\nu}(y) - u_a(x) \partial^\mu \tilde{u}_b(y)] \\ &+ d_2(x-y) h^{(4)}_{ab} \Phi_a(x) \partial^\mu \Phi_b(y) + d_2(x-y) h^{(5)}_{ab} \partial^\mu \Phi_a(x) \Phi_b(y) \\ &- \frac{i}{2}\, g^{(4)}_{ab}\, [\partial^\mu d_2(x-y) \Phi_a(x) \Phi_b(y) - d_2(x-y) \partial^\mu \Phi_a(x) \Phi_b(y)] \\ &- d_2(x-y) \bar{\Psi}(x) A_\epsilon \otimes \gamma^\mu \gamma_\epsilon \Psi(y) \end{aligned} \tag{4.23}$$

$$B^{\emptyset\emptyset}(x,y) = -d_2(x-y) h^{(3)}_{ab} [\partial_\mu \tilde{u}_a(x) v^\mu_b(y) - (x \leftrightarrow y)] \tag{4.24}$$

where

$$\begin{aligned} h^{(0)} = -\frac{i}{2}\tilde{g} \qquad & h^{(1)} = -i\left(\frac{1}{2}\tilde{g} + 3g\right) \\ h^{(2)} = -ig \qquad & h^{(3)} = i\left(\frac{1}{2}\tilde{g} + 2g\right) \\ h^{(4)} = ig^{(3)} \qquad & h^{(5)} = 2ig^{(3)} \end{aligned} \tag{4.25}$$

then we can prove the relation from the statement. As a result we have:

Theorem 4.3 *The one-loop contribution $D^{IJ}_{(1)}(x,y)$ is a coboundary, up to super-renormalizable terms. So the causal commutator $D^{IJ}(x,y)$ up to the second-order of the perturbation theory is the sum of the classical contribution (tree part) and quantum (loop) corrections which are super-renormalizable. This property remains true after causal decomposition, in particular for the chronological products.*

For the two-loop contribution in the second order of the perturbation theory we find the non-trivial expression

$$D^{\emptyset\emptyset}_{(2)0}(x,y) = ic\ \square d_3(x-y) \tag{4.26}$$

(where c is a constant). If we take

$$\begin{aligned} B^{\emptyset\emptyset}_{(2)}(x,y) &= 0 \\ B^{[\mu]\emptyset}_{(2)}(x,y) &= \frac{1}{2} c\ \partial^{\mu} d_3(x-y) \end{aligned} \tag{4.27}$$

then we can express the commutator $D^{\emptyset\emptyset}_{(2)0}(x,y)$ as a coboundary.

5 Gravity

We prove a result of the same nature for the extremely interesting case of massless gravity.

5.1 The Gravity Lagrangian

We have the following result [10].

Theorem 5.1 *Let T be a relative cocycle for d_Q which is tri-linear in the fields and is of canonical dimension $\omega(T) \leq 5$ and ghost number $gh(T) = 0$. Then: (i) T is (relatively) cohomologous to a non-trivial cocycle of the form:*

$$\begin{aligned} T = \kappa(2\ h_{\mu\rho}\ \partial^{\mu}h^{\nu\lambda}\ \partial^{\rho}h_{\nu\lambda} + 4\ h_{\nu\rho}\ \partial^{\lambda}h^{\mu\nu}\ \partial_{\mu}h_{\nu}{}^{\lambda} - 4\ h_{\rho\lambda}\ \partial^{\mu}h^{\nu\rho}\ \partial_{\mu}h_{\nu}{}^{\lambda} \\ +2\ h^{\rho\lambda}\ \partial_{\mu}h_{\rho\lambda}\ \partial^{\mu}h - h_{\mu\rho}\ \partial^{\mu}h\ d^{\rho}h - 4\ u^{\rho}\ \partial^{\nu}\tilde{u}^{\lambda}\ \partial_{\rho}h_{\nu\lambda} \\ +4\ \partial^{\rho}u^{\nu}\ \partial_{\nu}\tilde{u}^{\lambda}\ h_{\rho\lambda} + 4\ \partial^{\rho}u_{\nu}\ \partial^{\lambda}\tilde{u}_{\nu}\ h_{\rho\lambda} - 4\ \partial^{\nu}u_{\nu}\ \partial^{\rho}\tilde{u}^{\lambda}\ h_{\rho\lambda}) \end{aligned} \tag{5.1}$$

where $\kappa \in \mathbb{R}$.

(ii) The relation $d_QT = i\ \partial_{\mu}T^{\mu}$ is verified by:

$$\begin{aligned} T^{\mu} = \kappa(-2u^{\mu}\ \partial_{\nu}h_{\rho\lambda}\ \partial^{\rho}h^{\nu\lambda} + u^{\mu}\ \partial_{\rho}h_{\nu\lambda}\ \partial^{\rho}h^{\nu\lambda} - \frac{1}{2}u^{\mu}\ \partial_{\rho}h\ \partial^{\rho}h \\ +4\ u^{\rho}\ \partial^{\nu}h^{\mu\lambda}\ \partial_{\rho}h_{\nu\lambda} - 2\ u^{\rho}\ \partial^{\mu}h^{\nu\lambda}\ \partial_{\rho}h_{\nu\lambda} + u^{\rho}\ \partial^{\mu}h\ \partial_{\rho}h \\ -4\ \partial^{\rho}u^{\nu}\ \partial_{\nu}h^{\mu\lambda}\ h_{\rho\lambda} - 4\partial^{\ \rho}u_{\nu}\ \partial^{\lambda}h^{\mu\nu}\ h_{\rho\lambda} + 4\ \partial^{\lambda}u_{\rho}\ \partial^{\mu}h^{\nu\rho}\ h_{\nu\lambda} \\ +4\ \partial_{\nu}u^{\nu}\ \partial^{\rho}h^{\mu\lambda}\ h_{\rho\lambda} - 2\ \partial_{\nu}u^{\nu}\ \partial^{\mu}h^{\rho\lambda}\ h_{\rho\lambda} - 2\ \partial^{\rho}u^{\lambda}\ h_{\rho\lambda}\ \partial^{\mu}h + \partial^{\nu}u_{\nu}\ h\ \partial^{\mu}h \\ -2\ u^{\mu}\ \partial_{\nu}\partial_{\rho}u^{\rho}\ \tilde{u}^{\nu} + 2\ u_{\rho}\ \partial^{\rho}\partial^{\sigma}u_{\sigma}\ \tilde{u}^{\mu} - 2\ u^{\mu}\ \partial_{\lambda}u_{\rho}\ \partial^{\rho}\tilde{u}^{\nu} \\ +2\ u_{\rho}\ \partial_{\lambda}u^{\mu}\ \partial^{\rho}\tilde{u}^{\lambda} + 2\ \partial^{\rho}u_{\rho}\ \partial_{\lambda}u^{\mu}\ \tilde{u}^{\lambda} - 2\ u_{\rho}\ \partial^{\rho}u_{\lambda}\ \partial^{\mu}\tilde{u}^{\lambda}) \end{aligned} \tag{5.2}$$

(iii) The relation $d_QT^{\mu} = i\ \partial_{\nu}T^{\mu\nu}$ is verified by:

$$\begin{aligned} T^{\mu\nu} \equiv \kappa[2(-u^{\mu}\ \partial_{\lambda}u_{\rho}\ \partial^{\rho}h^{\nu\lambda} + u_{\rho}\ \partial_{\lambda}u^{\mu}\ \partial^{\rho}h^{\nu\lambda} + u_{\rho}\ \partial^{\rho}u_{\lambda}\ \partial^{\nu}h^{\mu\lambda} + \partial_{\rho}u^{\rho}\ \partial_{\lambda}u^{\mu}\ h^{\nu\lambda}) \\ -(\mu \leftrightarrow \nu) + 4\ \partial^{\lambda}u^{\mu}\ \partial^{\rho}u^{\nu}\ h_{\rho\lambda}]. \end{aligned} \tag{5.3}$$

(iv) The relation $d_QT^{\mu\nu} = i\ \partial_{\rho}T^{\mu\nu\rho}$ is verified by:

$$T^{\mu\nu\rho} \equiv \kappa[2u_{\lambda}\ \partial^{\lambda}u^{\rho}\ u^{\mu\nu} - u_{\rho}\ (\partial^{\mu}u^{\lambda}\ \partial_{\lambda}u^{\nu} - \partial^{\nu}u^{\lambda}\ \partial_{\lambda}u^{\mu}) + \text{circular perm.}] \tag{5.4}$$

and we have $d_QT^{\mu\nu\rho} = 0$.

(v) The cocycles $T, T^{\mu}, T^{\mu\nu}$ and $T^{\mu\nu\rho}$ are non-trivial and invariant with respect to parity. Here

$$u_{\mu\nu} \equiv \partial_{\mu}u_{\nu} - \partial_{\nu}u_{\mu}. \tag{5.5}$$

5.2 The Generic Expressions for the One-Loop Cochains

We consider the one-loop contribution $D^{IJ}_{(1)}(x,y)$ and we do not need the splitting (4.11) of the Pauli-Jordan causal commutator from the preceding Section because the mass is already null. There is a particularity of the gravity case, namely we do not need to compute explicitly these expressions. The result is of pure cohomology nature. We only need to provide a generic expression for $D^{IJ}_{(1)}(x,y)$ and impose the cochain condition

$$sD = 0 \tag{5.6}$$

which follows from the gauge invariance of the interaction Lagrangian $sT = 0$. A number of limitations will result on the various arbitrary coefficients and we will be able to prove that D is a coboundary. We give only the relevant coefficients. For instance from the expression $D^{[\mu]\emptyset}$ we need only:

$$\begin{aligned}
D^{[\mu]\emptyset}(x,y) = \cdots + F_3\ \partial^\mu\partial^\nu\Box d_2(x-y)\ u^\rho(x)\ h_{\nu\rho}(y) + \cdots \\
+F_{12}\ \partial_\rho\partial_\sigma\Box d_2(x-y)\ h^{\mu\sigma}(x)\ u^\rho(y) \\
+F_{13}\ \Box^2 d_2(x-y)\ u_\nu(x)\ h^{\mu\nu}(y) + F_{14}\ \Box^2 d_2(x-y)\ h^{\mu\nu}(x)\ u_\nu(y)
\end{aligned} \tag{5.7}$$

From the expression $D^{[\mu][\nu]}$ we need the whole sector

$$\begin{aligned}
D^{[\mu][\nu]}_1(x,y) = K_1\ \partial^\mu\partial^\nu\partial^\rho\partial^\sigma d_2(x-y)\ u_\rho(x)\ u_\sigma(y) \\
+ K_2\ \partial^\mu\partial^\nu\Box d_2(x-y)\ u_\rho(x)\ u^\rho(y) \\
+ K_3\ [\partial^\mu\partial^\rho\Box d_2(x-y)\ u_\rho(x)\ u^\nu(y) + \partial^\nu\partial^\rho\ \Box d_2(x-y)\ u^\mu(x)\ u_\rho(y)] \\
+ K_4\ [\partial^\mu\partial^\rho\Box d_2(x-y)\ u^\nu(x)\ u_\rho(y) + \partial^\nu\partial^\rho\ \Box d_2(x-y)\ u_\rho(x)\ u^\mu(y)] \\
+ K_5\ \eta^{\mu\nu}\ \partial^\rho\partial^\sigma\Box d_2(x-y)\ u_\rho(x)\ u_\sigma(y) \\
+ K_6\ \eta^{\mu\nu}\ \Box^2 d_2(x-y)\ u_\rho(x)\ u^\rho(y) \\
+K_7\ \Box^2 d_2(x-y)\ u^\mu(x)\ u^\nu(y) \\
+K_8\ \Box^2 d_2(x-y)\ u^\nu(x)\ u^\mu(y)
\end{aligned} \tag{5.8}$$

and only a few terms from the sector:

$$\begin{aligned}
D^{[\mu][\nu]}_2(x,y) = \cdots + L_2\ \partial^\mu\partial^\nu\partial^\rho d_2(x-y)\ [u^\sigma(x)\ \partial_\rho u_\sigma(y) - \partial_\rho u_\sigma(x)\ u^\sigma(y)] \\
+L_3\ \partial^\mu\partial^\nu\partial^\rho d_2(x-y)\ [u^\sigma(x)\ \partial_\sigma u_\rho(y) - \partial_\sigma u_\rho(x)\ u^\sigma(y)] + \cdots \\
+L_5\ [\partial^\mu\partial^\rho\partial^\sigma d_2(x-y)\ u_\rho(x)\ \partial^\nu u_\sigma(y) - \partial^\nu\partial^\rho\partial^\sigma d_2(x-y)\ \partial^\mu u_\sigma(x)\ u_\rho(y)] \\
+L_6\ [\partial^\mu\partial^\rho\partial^\sigma d_2(x-y)\ u_\rho(x)\ \partial_\sigma u^\nu(y) - \partial^\nu\partial^\rho\partial^\sigma d_2(x-y)\ \partial_\sigma u^\mu(x)\ u_\rho(y)] + \cdots \\
+L_8\ [\partial^\mu\Box d_2(x-y)\ u_\rho(x)\ \partial^\nu u^\rho(y) - \partial^\nu\ \Box d_2(x-y)\ \partial^\mu u^\rho(x)\ u_\rho(y)] \\
+ L_9\ [\partial^\mu\Box d_2(x-y)\ u_\rho(x)\ \partial^\rho u^\nu(y) - \partial^\nu\ \Box d_2(x-y)\ \partial^\rho u^\mu(x)\ u_\rho(y)] + \\
+L_{10}\ \partial_\rho\Box d_2(x-y)\ [u^\rho(x)\ \partial^\mu u^\nu(y) - \partial^\nu u^\mu(x)\ u_\rho(y)] \\
+L_{11}\ \partial_\rho\Box d_2(x-y)\ [u^\rho(x)\ \partial^\nu u^\mu(y) - \partial^\mu u^\nu(x)\ u_\rho(y)] \\
+L_{12}\ \partial_\rho\Box d_2(x-y)\ [u^\mu(x)\ \partial^\rho u^\nu(y) - \partial^\rho u^\mu(x)\ u_\nu(y)] \\
+L_{13}\ \partial_\rho\Box d_2(x-y)\ [u^\mu(x)\ \partial^\nu u^\rho(y) - \partial^\mu u^\rho(x)\ u_\nu(y)] + \cdots
\end{aligned}$$

$$+L_{18}\ \eta^{\mu\nu}\ \partial^\rho\Box d_2(x-y)\ [u^\sigma(x)\ \partial_\rho u_\sigma(y) - \partial_\rho u_\sigma(x)\ u^\sigma(y)]$$
$$+L_{19}\ \eta^{\mu\nu}\ \partial^\rho\Box d_2(x-y)\ [u^\sigma(x)\ \partial_\sigma u_\rho(y) - \partial_\sigma u_\rho(x)\ u^\sigma(y)] + \cdots$$
$$+ L_{24}\ [\partial^\nu\Box d_2(x-y)\ u_\rho(x)\ \partial^\mu u^\rho(y) - \partial^\mu\ \Box d_2(x-y)\ \partial^\nu u^\rho(x)\ u_\rho(y)]$$
$$+ L_{25}\ [\partial^\nu\Box d_2(x-y)\ u_\rho(x)\ \partial^\rho u^\mu(y) - \partial^\mu\ \Box d_2(x-y)\ \partial^\rho u^\nu(x)\ u_\rho(y)] + \cdots \tag{5.9}$$

From the expression $D^{[\mu\nu]\emptyset}$ we need the whole sector

$$\begin{aligned} D_1^{[\mu\nu]\emptyset}(x,y) = {} & Q_1\ [\partial^\mu\partial^\rho\Box d_2(x-y)\ u^\nu(x)\ u_\rho(y) - (\mu\leftrightarrow\nu)] \\ & +Q_2\ [\partial^\mu\partial^\rho\Box d_2(x-y)\ u_\rho(x)\ u^\nu(y) - (\mu\leftrightarrow\nu)] \\ & +Q_3\ \Box^2 d_2(x-y)\ [u^\mu(x)\ u^\nu(y) - (\mu\leftrightarrow\nu)] \end{aligned} \tag{5.10}$$

and some terms from the sector

$$\begin{aligned} D_2^{[\mu\nu]\emptyset}(x,y) = {} & \cdots + R_5\ [\partial^\mu\Box d_2(x-y)\ u_\rho(x)\ \partial^\nu u^\rho(y) - (\mu\leftrightarrow\nu)] \\ & + R_6\ [\partial^\mu\Box d_2(x-y)\ u_\rho(x)\ \partial^\rho u^\nu(y) - (\mu\leftrightarrow\nu)] + \cdots \end{aligned} \tag{5.11}$$

and from

$$\begin{aligned} D_3^{[\mu\nu]\emptyset}(x,y) = {} & \cdots + S_2\ [\partial^\mu\partial^\rho\partial^\sigma d_2(x-y)\ \partial^\nu u_\rho(x)\ u_\sigma(y) - (\mu\leftrightarrow\nu)] \\ & +S_3\ [\partial^\mu\partial^\rho\partial^\sigma d_2(x-y)\ \partial_\rho u^\nu(x)\ u_\sigma(y) - (\mu\leftrightarrow\nu)] + \cdots \\ & +S_5\ [\partial^\mu\Box d_2(x-y)\ \partial^\nu u^\rho(x)\ u_\rho(y) - (\mu\leftrightarrow\nu)] \\ & +S_6\ [\partial^\mu\Box d_2(x-y)\ \partial^\rho u^\nu(x)\ u_\rho(y) - (\mu\leftrightarrow\nu) \\ & +S_7\ \partial_\rho\Box d_2(x-y)\ [\partial^\mu u^\nu(x) - (\mu\leftrightarrow\nu)]\ u^\rho(y) \\ & +S_8\ \partial_\rho\Box d_2(x-y)\ \partial^\rho u^\mu(x)\ u^\nu(y) - (\mu\leftrightarrow\nu) \\ & +S_9\ \partial_\rho\Box d_2(x-y)\ \partial^\mu u^\rho(x)\ u^\nu(y) - (\mu\leftrightarrow\nu) \end{aligned} \tag{5.12}$$

5.3 Relative Cocycle Equations

Now we consider the cocycle equation

$$sD = 0 \qquad \Leftrightarrow \qquad d_Q D^{IJ} = i\frac{\partial}{\partial x^\rho} D^{I\rho,J} + i(-1)^{|I|}\frac{\partial}{\partial y^\rho} D^{I,J\rho}. \tag{5.13}$$

From

$$d_Q D^{[\mu]\emptyset} = i\frac{\partial}{\partial x^\nu} D^{[\mu\nu]\emptyset} - i\frac{\partial}{\partial y^\nu} D^{[\mu][\nu]} \tag{5.14}$$

we obtain:

- from the coefficients of the monomials $\partial\partial\partial\partial d_2(x-y)u(x)u(y)$

$$K_1 + K_3 + K_4 + K_5 + Q_1 + Q_2 = 0 \tag{5.15}$$

$$K_2 + K_6 = 0 \tag{5.16}$$

$$K_3 + K_7 - Q_1 + Q_3 = 0 \tag{5.17}$$

$$K_4 + K_8 - Q_2 - Q_3 = 0 \tag{5.18}$$

From (5.15) + (5.17) + (5.18) we obtain

$$K_1 + 2K_3 + 2K_4 + K_5 + K_7 + K_8 = 0 \tag{5.19}$$

- from the coefficients of the monomials $\partial\partial\partial\partial d_2(x-y)u(x)\partial u(y)$

$$-K_2 + L_2 + L_8 + L_{18} + R_5 = \frac{1}{2}\ F_3 \tag{5.20}$$

$$-K_4 + L_3 + L_9 + L_{19} + R_6 = \frac{1}{2}\ F_3 \tag{5.21}$$

$$-K_6 + L_{24} - R_5 = \frac{1}{2}\ F_{13} \tag{5.22}$$

$$-K_8 + L_{25} - R_6 = \frac{1}{2}\ F_{13} \tag{5.23}$$

Taking the difference we obtain

$$-K_2 + K_4 + L_2 - L_3 + L_8 - L_9 + L_{18} - L_{19} + R_5 - R_6 = 0 \tag{5.24}$$

$$K_6 - K_8 - L_{24} + L_{25} + R_5 - R_6 = 0 \tag{5.25}$$

If we subtract these equations and use (5.16) we get

$$K_4 + K_8 + L_2 - L_3 + L_8 - L_9 + L_{18} - L_{19} + L_{24} - L_{25} = 0 \tag{5.26}$$

- from the coefficients of the monomials $\partial\partial\partial\partial d_2(x-y)\partial u(x)u(y)$

$$-L_5 - L_{11} - L_{13} - S_2 + S_7 + S_9 = -\frac{1}{2}\ F_{12} \tag{5.27}$$

$$-L_6 - L_{10} - L_{12} - Q_1 - S_3 - S_7 + S_8 = -\frac{1}{2}\ F_{12} \tag{5.28}$$

$$-L_8 - S_5 = -\frac{1}{2}\ F_{14} \tag{5.29}$$

$$-L_9 + Q_3 - S_6 = -\frac{1}{2}\ F_{14} \tag{5.30}$$

Taking the difference we obtain

$$-L_5 + L_6 + L_{10} - L_{11} + L_{12} - L_{13} + Q_1 - S_2 + S_3 + 2S_7 - S_8 + S_9 = 0 \tag{5.31}$$

$$L_8 - L_9 + Q_3 + S_5 - S_6 = 0 \tag{5.32}$$

If we add the first equation with (5.17) and (5.18) we obtain

$$K_3 + K_4 + K_7 + K_8 - L_5 + L_6 + L_{10} - L_{11} + L_{12} - L_{13} - Q_2 - S_2 + S_3 + 2S_7 - S_8 + S_9 = 0 \tag{5.33}$$

and if we add the second equation with (5.18) we obtain

$$K_4 + K_8 + L_8 - L_9 - Q_2 + S_5 - S_6 = 0. \tag{5.34}$$

5.4 The Generic Expressions for the Coboundaries B^{IJ}

Theorem 5.2 *The coboundary equation*

$$D^{IJ} = (\bar{s}B)^{IJ}, \qquad |I| + |J| = 2 \tag{5.35}$$

is true iff the coefficients of the left hand side verify:

$$\begin{aligned} K_1 + 2K_3 + 2K_4 + K_5 + K_7 + K_8 = 0 \\ K_2 + K_6 = 0 \\ K_4 + K_8 - Q_2 - Q_3 = 0 \\ K_3 + K_7 - Q_1 + Q_3 = 0 \\ K_6 - K_8 - L_{24} + L_{25} + R_5 - R_6 = 0 \\ K_4 + K_8 + L_8 - L_9 - Q_2 + S_5 - S_6 = 0 \\ K_4 + K_8 + L_2 - L_3 + L_8 - L_9 + L_{18} - L_{19} + L_{24} - L_{25} = 0 \\ K_3 + K_4 + K_7 + K_8 - L_5 + L_6 + L_{10} - L_{11} + L_{12} - L_{13} \\ -Q_2 - S_2 + S_3 + 2S_7 - S_8 + S_9 = 0. \end{aligned} \tag{5.36}$$

Proof: (i) We need the generic form of the cocycles B^{IJ} constrained by

$$gh(B^{IJ}) = |I| + |J| - 1, \qquad \omega(B^{IJ}) = 5. \tag{5.37}$$

We will give only a number of relevant terms:

$$\begin{aligned} B_1^{[\mu\nu][\rho]}(x,y) = a_1[\partial^\mu\partial^\rho\partial^\sigma d_2(x-y)\ u^\nu(x)\ u_\sigma(y) - (\mu \leftrightarrow \nu)] \\ + a_2\ [\partial^\mu\partial^\rho\partial^\sigma d_2(x-y)\ u_\sigma(x)\ u^\nu(y) - (\mu \leftrightarrow \nu)] \\ + a_3\ [\partial^\mu \Box d_2(x-y)\ u^\nu(x)\ u^\rho(y) - (\mu \leftrightarrow \nu)] \\ + a_4\ [\partial^\mu \Box d_2(x-y)\ u^\rho(x)\ u^\nu(y) - (\mu \leftrightarrow \nu)] \\ + a_5\ [\partial^\rho \Box d_2(x-y)\ u^\mu(x)\ u^\nu(y) - (\mu \leftrightarrow \nu)] \\ + a_6\ [\eta^{\mu\rho}\ \partial^\nu \Box d_2(x-y) u_\sigma(x)\ u^\sigma(y) - (\mu \leftrightarrow \nu)] \\ + a_7\ [\eta^{\mu\rho}\ \partial^\sigma \Box d_2(x-y) u^\nu(x)\ u_\sigma(y) - (\mu \leftrightarrow \nu)] \\ + a_8\ [\eta^{\mu\rho}\ \partial^\sigma \Box d_2(x-y) u_\sigma(x)\ u^\nu(y) - (\mu \leftrightarrow \nu)] \\ + a_9\ [\eta^{\mu\rho}\ \partial^\nu\partial^\sigma\partial^\lambda d_2(x-y) u_\sigma(x)\ u_\lambda(y) - (\mu \leftrightarrow \nu)] \end{aligned} \tag{5.38}$$

$$\begin{aligned} B_2^{[\mu\nu][\rho]}(x,y) = b_1[\partial^\mu\partial^\rho d_2(x-y)\ \partial^\nu u^\sigma(x)\ u_\sigma(y) - (\mu \leftrightarrow \nu)] \\ + b_2[\partial^\mu\partial^\rho d_2(x-y)\ u_\sigma(x)\ \partial^\nu u^\sigma(y) - (\mu \leftrightarrow \nu)] \\ + b_3\ [\partial^\mu\partial^\rho d_2(x-y)\ \partial^\sigma u^\nu(x)\ u_\sigma(y) - (\mu \leftrightarrow \nu)] \\ + b_4\ [\partial^\mu\partial^\rho d_2(x-y)\ u_\sigma(x)\ \partial^\sigma u^\nu(y) - (\mu \leftrightarrow \nu)] + \cdots \\ + b_7\ [\partial^\mu\partial^\sigma d_2(x-y)\ \partial^\nu u^\rho(x)\ u_\sigma(y) - (\mu \leftrightarrow \nu)] + \cdots \end{aligned}$$

$$
\begin{aligned}
&+b_9\ [\partial^\mu\partial^\sigma d_2(x-y)\ \partial^\nu u_\sigma(x)\ u^\rho(y)-(\mu\leftrightarrow\nu)]\\
&+b_{11}\ [\partial^\mu\partial^\sigma d_2(x-y)\ \partial^\rho u^\nu(x)\ u_\sigma(y)-(\mu\leftrightarrow\nu)]\\
&+b_{13}\ [\partial^\mu\partial^\sigma d_2(x-y)\ \partial_\sigma u^\nu(x)\ u^\rho(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+b_{19}\ \partial^\rho\partial^\sigma d_2(x-y)\ [\partial^\mu u^\nu(x)\ u_\sigma(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+b_{21}\ \partial^\rho\partial^\sigma d_2(x-y)\ [\partial^\mu u_\sigma(x)\ u^\nu(y)-(\mu\leftrightarrow\nu)]\\
&+b_{22}\ \partial^\rho\partial^\sigma d_2(x-y)\ [u^\mu(x)\ \partial^\nu u_\sigma(y)-(\mu\leftrightarrow\nu)]\\
&+b_{23}\ \partial^\rho\partial^\sigma d_2(x-y)\ [\partial_\sigma u^\mu(x)\ u^\nu(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+b_{25}\ \Box d_2(x-y)\ [\partial^\mu u^\nu(x)\ u^\rho(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+b_{27}\ \Box d_2(x-y)\ [\partial^\mu u^\rho(x)\ u^\nu(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+b_{29}\ \Box d_2(x-y)\ [\partial^\rho u^\mu(x)\ u^\nu(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+b_{31}\ [\eta^{\mu\rho}\ \partial^\nu\partial^\sigma d_2(x-y)\ \partial_\sigma u_\lambda(x)\ u^\lambda(y)-(\mu\leftrightarrow\nu)]\\
&+b_{32}\ [\eta^{\mu\rho}\ \partial^\nu\partial^\sigma d_2(x-y)\ u^\lambda(x)\ \partial_\sigma u_\lambda(y)-(\mu\leftrightarrow\nu)]\\
&+b_{33}\ [\eta^{\mu\rho}\ \partial^\nu\partial^\sigma d_2(x-y)\ \partial_\lambda u_\sigma(x)\ u^\lambda(y)-(\mu\leftrightarrow\nu)]\\
&+b_{34}\ [\eta^{\mu\rho}\ \partial^\nu\partial^\sigma d_2(x-y)\ u^\lambda(x)\ \partial_\lambda u_\sigma(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+b_{37}\ \partial^\sigma\partial^\lambda d_2(x-y)\ [\eta^{\mu\rho}\ \ \partial^\nu u_\sigma(x)\ u_\lambda(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+b_{39}\ \partial^\sigma\partial^\lambda d_2(x-y)\ [\eta^{\mu\rho}\ \ \partial_\sigma u^\nu(x)\ u_\lambda(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+b_{43}\ \Box d_2(x-y)\ [\eta^{\mu\rho}\ \partial^\nu u^\sigma(x)\ u_\sigma(y)-(\mu\leftrightarrow\nu)]\\
&+b_{44}\ \Box d_2(x-y)\ [\eta^{\mu\rho}\ u_\sigma(x)\ \partial^\nu u^\sigma(y)-(\mu\leftrightarrow\nu)]\\
&+b_{45}\ \Box d_2(x-y)\ [\eta^{\mu\rho}\ \partial^\sigma u^\nu(x)\ u_\sigma(y)-(\mu\leftrightarrow\nu)]\\
&+b_{46}\ \Box d_2(x-y)\ [\eta^{\mu\rho}\ u_\sigma(x)\ \partial^\sigma u^\nu(y)-(\mu\leftrightarrow\nu)]+\cdots
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
B_1^{[\mu\nu]\emptyset}(x,y)=\cdots&+g_4\ [\partial^\mu\partial_\rho\partial_\sigma d_2(x-y)\ h^{\nu\rho}(x)\ u^\sigma(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+g_7[\partial^\mu\Box d_2(x-y)\ u_\rho(x)\ h^{\nu\rho}(y)-(\mu\leftrightarrow\nu)]\\
&+g_8\ [\partial^\mu\Box d_2(x-y)\ h^{\nu\rho}(x)\ u_\rho(y)-(\mu\leftrightarrow\nu)]+\cdots\\
&+g_{10}\ \partial_\rho\Box d_2(x-y)\ h^{\mu\rho}(x)\ u^\nu(y)-(\mu\leftrightarrow\nu)]
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
B_1^{[\mu][\nu]}(x,y)=\cdots&+r_2\ \partial^\mu\partial^\nu\partial^\rho d_2(x-y)\ [u^\sigma(x)\ h_{\rho\sigma}(y)+h_{\rho\sigma}(x)\ u^\sigma(y)]+\cdots\\
&+r_8\ [\partial^\mu\Box d_2(x-y)\ u_\rho(x)\ h^{\nu\rho}(y)+\partial^\nu\Box\ d_2(x-y)\ h^{\mu\rho}(x)\ u_\rho(y)]\\
&+r_{10}\ [\partial^\nu\Box d_2(x-y)\ u_\rho(x)\ h^{\mu\rho}(x)+\partial^\mu\Box\ d_2(x-y)\ h^{\nu\rho}(x)\ u_\rho(y)]+\cdots\\
&+r_{16}\ \eta^{\mu\nu}\ \partial^\rho\Box d_2(x-y)\ [u^\sigma(x)\ h_{\rho\sigma}(x)+h_{\rho\sigma}(x)\ u^\sigma(y)]
\end{aligned}
\tag{5.41}
$$

(ii) From the relation

$$
\begin{aligned}
&(\bar{s}B)^{[\mu][\nu]}=D^{[\mu][\nu]}\qquad\Leftrightarrow\\
&\frac{\partial}{\partial x^\rho}B^{[\mu\rho][\nu]}+(x\leftrightarrow y,\mu\leftrightarrow\nu)-id_QB^{[\mu][\nu]}=-iD^{[\mu][\nu]}
\end{aligned}
\tag{5.42}
$$

we obtain many equations; we only select:

$$\begin{aligned}
2a_1 + 2a_3 - 2a_9 &= -iK_1 \\
-a_1 + a_3 + a_5 - a_7 &= -iK_3 \\
-a_2 + a_4 - a_5 - a_8 &= -iK_4 \\
-2a_3 &= -iK_4 \\
-2a_4 &= -iK_8 \\
2a_6 &= -iK_6 \\
-2a_6 &= -iK_2 \\
2a_7 + 2a_8 + 2a_9 &= -iK_5 \\
a_2 + b_3 - b_{33} - b_4 + b_{34} - \frac{1}{2}r_2 &= iL_3 \\
a_4 + b_4 + b_{46} - \frac{1}{2}r_{10} &= iL_{25} \\
a_5 - b_3 - b_{45} - \frac{1}{2}r_8 &= iL_9 \\
a_6 + b_{31} + b_{43} - b_{32} - b_{44} - \frac{1}{2}r_{16} &= iL_{18} \\
-a_6 + b_2 + b_{44} - \frac{1}{2}r_{10} &= iL_{24} \\
a_8 + b_{33} + b_{45} - b_{34} - b_{46} - \frac{1}{2}r_{16} &= iL_{19} \\
b_1 - b_{31} - b_2 + b_{32} - \frac{1}{2}r_2 &= iL_2 \\
-b_1 - b_{43} - \frac{1}{2}r_8 &= iL_8
\end{aligned} \tag{5.43}$$

(iii) From the relation

$$\begin{aligned}
(\bar{s}B)^{[\mu\nu]\emptyset} &= D^{[\mu\nu]\emptyset} \qquad \Leftrightarrow \\
\frac{\partial}{\partial y^\rho}B^{[\mu\nu][\rho]} - id_Q B^{[\mu\nu]\emptyset} &= -iD^{[\mu\nu]\emptyset}
\end{aligned} \tag{5.44}$$

we obtain as above many equations and we select:

$$\begin{aligned}
-\,a_1 - a_3 - a_7 &= -iQ_1 \\
-a_2 - a_4 - a_8 &= -iQ_2 \\
-a_5 &= -iQ_3 \\
a_4 - b_4 - b_{46} + \frac{1}{2}g_7 &= -iR_6 \\
-a_6 - b_2 - b_{44} + \frac{1}{2}g_7 &= -iR_5 \\
-b_1 - b_{43} - \frac{1}{2}g_8 &= -iS_5
\end{aligned}$$

$$\begin{aligned}
-b_3 - b_{45} - \frac{1}{2} g_8 &= -iS_6 \\
-b_7 - b_9 - b_{37} - \frac{1}{2} g_4 &= -iS_2 \\
-b_{11} - b_{13} - b_{39} - \frac{1}{2} g_4 &= -iS_3 \\
-b_{19} - b_{25} &= -iS_7 \\
-b_{21} - b_{27} - \frac{1}{2} g_{10} &= -iS_9 \\
-b_{22} - b_{29} - \frac{1}{2} g_{10} &= -iS_8.
\end{aligned} \tag{5.45}$$

(iv) Now we can show that the preceding systems are compatible *iff* we have the equations from the statement. It can be proved by direct computations that no other equations are needed to obtain a solution of the coboundary equation from the statement. This assertion follows from hard work: one has to write down the generic expressions for the coboundaries $B^{IJ}, |I| + |J| =, 2, 3$ and show that a solution of the equations (5.42) and (5.44) exists *iff* the eight equations from the statement are true. ■

Now we notice that from the relative cocycle equations we have obtained (5.19), (5.16), (5.18), (5.17), (5.25), (5.34), (5.26) and (5.33) which are exactly the equations from the statement of the theorem. As a conclusion, we have

Corollary 5.3 *If D^{IJ} is a cocycle, then we can write $D^{IJ}, |I| + |J| = 2$ as a coboundary.*

5.5 The Descent Procedure

We now start a descent procedure. If we use the preceding corollary in the cocycle equation

$$(sD)^{[\mu]\emptyset} = 0 \tag{5.46}$$

we obtain that the expression

$$\tilde{D}^{[\mu]\emptyset} \equiv D^{[\mu]\emptyset} - i\frac{\partial}{\partial x^\nu} B^{[\mu\nu]\emptyset} + i\frac{\partial}{\partial y^\nu} B^{[\mu][\nu]} \tag{5.47}$$

is a cocycle

$$d_Q \tilde{D}^{[\mu]\emptyset} = 0. \tag{5.48}$$

Using this cocycle equation one can prove that $\tilde{D}^{[\mu]\emptyset}$ is in fact a coboundary. For this one must consider all relevant sectors of this expression. In the sectors

$$\begin{aligned}
&\partial\partial\partial\partial d_2(x-y)[u(x)\ h(y) + h(x)\ u(y)] \\
&\partial\partial\partial\partial d_2(x-y)[u(x)\ \partial h(y) + \partial h(x)\ u(y)] \\
&\partial\partial\partial\partial d_2(x-y)[\partial u(x)\ \partial h(y) + \partial h(x)\ \partial u(y)]
\end{aligned}$$

this result follows elementary. In the sector

$$\partial\partial\partial d_2(x-y)[\partial u(x)\ h(y) + h(x)\ \partial u(y)]$$

we are left with 11 nontrivial cocycles, some of which cannot be seen immediatley as coboundaries. For instance

$$\begin{aligned} D^{[\mu]\emptyset}(x,y) = \partial^\nu \Box d_2(x-y)[h^{\mu\rho}(x)\partial_\nu u_\rho(y) + h^{\mu\rho}(x)\partial_\rho u_\nu(y) \\ +\partial^\mu u^\rho(x)h_{\nu\rho}(y) + \partial^\rho u^\mu(x)h_{\nu\rho}(y) - \partial^\rho u_\rho(x)h^\mu{}_\nu(y) \\ -\frac{1}{2}\partial_\nu u^\mu(x)h(y) - \frac{1}{2}\partial^\mu u_\nu(x)h(y) + \frac{1}{2}\delta^\mu_\nu\partial^\rho u_\rho(x)h(y)] \end{aligned}$$

can be written as

$$i\partial^\nu \Box d_2(x-y)d_Q[2h^{\mu\rho}(x)h_{\nu\rho}(y) - h^\mu{}_\nu(x)h(y)]. \tag{5.49}$$

The sector

$$\partial d_2(x-y)[\partial\partial\partial u(x)\ \partial h(y) + \partial h(x)\ \partial\partial\partial u(y)]$$

is the most complicated one. There are 12 cocycles in which ∂h appears in the combination $\partial_\sigma h^{\rho\sigma}$ so these are seen immediately to be coboundaries. But we are still left with 9 nontrivial cocycles as for instance

$$\begin{aligned} D^{[\mu]\emptyset}(x,y) = \partial^\nu d_2(x-y)[\partial_\nu h_{\rho\sigma}(x)\partial^\mu\partial^\rho u^\sigma(y) + \partial_\nu\partial_\rho u_\sigma(x)\partial^\mu h^{\rho\sigma}(y)] \\ -\partial^\mu d_2(x-y)[\partial^\nu h^{\rho\sigma}(x)\partial_\mu\partial_\rho u_\sigma(y) + \partial_\nu\partial_\rho u_\sigma(x)\partial^\nu h^{\rho\sigma}(y)] \end{aligned}$$

which can be written as

$$\begin{aligned} i\partial^\nu d_2(x-y)d_Q\left[\partial_\nu h_{\rho\sigma}(x)\partial^\mu h^{\rho\sigma}(y) - \frac{1}{2}\partial_\nu h(x)\partial^\mu(y)\right] \\ -i\partial^\mu d_2(x-y)d_Q\left[\partial_\nu h_{\rho\sigma}(x)\partial^\nu h^{\rho\sigma}(y) - \frac{1}{2}\partial_\nu h(x)\partial^\nu(y)\right] \end{aligned}$$

In the end we prove that

$$\tilde{D}^{[\mu]\emptyset} = d_Q B^{[\mu]\emptyset} \tag{5.50}$$

so we obtain

$$D^{[\mu]\emptyset} = d_Q B^{[\mu]\emptyset} + i\frac{\partial}{\partial x^\nu}B^{[\mu\nu]\emptyset} - i\frac{\partial}{\partial y^\nu}B^{[\mu][\nu]} \tag{5.51}$$

i.e. the expression $D^{[\mu]\emptyset}$ is a relative coboundary.

We insert this result in the cocycle equation

$$(sD)^{\emptyset\emptyset} = 0 \tag{5.52}$$

and we obtain that the expression

$$\tilde{D}^\emptyset = D^{\emptyset\emptyset} - i\frac{\partial}{\partial x^\nu}B^{[\mu]\emptyset} - i\frac{\partial}{\partial y^\nu}B^{\emptyset[\mu]} \tag{5.53}$$

is a coboundary

$$d_Q\ \tilde{D}^\emptyset = 0. \tag{5.54}$$

If we write the generic form of $\tilde{D}^{\emptyset\emptyset}$ we can prove rather easy that in fact the preceding equation leads to

$$\tilde{D}^{\emptyset\emptyset} = 0 \tag{5.55}$$

so we have

$$D^{\emptyset\emptyset} = i\frac{\partial}{\partial x^{\nu}} B^{[\mu]\emptyset} + i\frac{\partial}{\partial y^{\nu}} B^{\emptyset[\mu]} \tag{5.56}$$

i.e. a relative cocycle if we take

$$B^{\emptyset\emptyset} = 0. \tag{5.57}$$

So we have proved the triviality of the cohomology problem. It is important to stress again that it was not necessary to compute explicitly the expressions D^{IJ}. In the end we have

Theorem 5.4 *For the pure gravity case, let us consider the expressions $D^{IJ}(x,y)$, up to the second order of the perturbation theory. Then these are cohomologous to the tree contribution i.e. the loop contribution is trivial.*

Proof: The preceding cohomologous argument has proved the assertion for one-loop contributions. For two-loop we have by direct computation the following non-trivial contribution:

$$D^{\emptyset\emptyset}_{(2)}(x,y) = ic\, \square^2 d_3(x-y) \tag{5.58}$$

(where c is some constant). If we take

$$\begin{aligned} B^{\emptyset\emptyset}_{(2)}(x,y) &= 0 \\ B^{[\mu]\emptyset}_{(2)}(x,y) &= \frac{1}{2} c\ \partial^{\mu} \square d_3(x-y) \end{aligned} \tag{5.59}$$

then we can write the two-loop contribution as a coboundary.

This means that, up to the second order of the perturbation theory, pure gravity is a classical theory. We can consider in the same way the loop contributions coming from the interaction between Yang-Mills and gravity. One can prove in fact that the cocycle equation forces this loop contribution to be null.

6 Conclusions

We have proved that the loop contributions to the causal commutator $D^{IJ}_{(1)}$ are of the form $sB+$ super-renormalizable terms in the Yang-Mills case and simply of the form sB in the pure gravity case. Because the expressions B have also causal support this property stays true after causal splitting. If

$$B^{IJ} = B^{IJ,\mathrm{adv}} - B^{IJ,ret} \tag{6.1}$$

is a causal splitting, then we have

$$A^{IJ}_{(1)} = sB^{\mathrm{adv}} + \mathrm{super} - \mathrm{renormalizable\ terms} \tag{6.2}$$

This means that the main contributions of the perturbation theory are the tree contributions which correspond to the classical theory. The quantum corrections associated to the loop graphs are behaving better in the ultra-violet limit; we conjecture that this result stays true in all orders of the perturbation theory. So there is a chance to construct a non-perturbative theory for gauge models. This follows from the well-known fact that the construction of non-trivial QFT models in $1+2$ and $1+1$ dimensions is closely connected to the super-renormalizability of the associated perturbation theory. This means that gauge models are better than say, the Φ^4 model in 4 dimensions, for which it is conjectured that the constructive quantum field theory does not exists. This is related to the fact that the Φ^4 model in 4 dimensions is only renormalizable (does not have any super-renormalizable properties for the loop contributions).

The preceding ideas are a full program for a new line of analysis of quantum field theories. We will continue in another paper with the much modest problem of investigating the conjecture in third order of the perturbation theory.

References

[1] N. N. Bogoliubov, D. Shirkov, "*Introduction to the Theory of Quantized Fields*", John Wiley and Sons, 1976 (3rd edition)

[2] M. Dütsch, K. Fredenhagen, "*A Local (Perturbative) Construction of Observables in Gauge Theories: the Example of QED*", Commun. Math. Phys. **203** (1999) 71-105

[3] M. Dütsch, K. Fredenhagen, "*Algebraic Quantum Field Theory, Perturbation Theory, and the Loop Expansion*", Commun.Math.Phys. **219** (2001) 5-30

[4] H. Epstein, V. Glaser, "*The Rôle of Locality in Perturbation Theory*", Ann. Inst. H. Poincaré **19 A** (1973) 211-295

[5] V. Glaser, "*Electrodynamique Quantique*", L'enseignement du 3e cycle de la physique en Suisse Romande (CICP), Semestre d'hiver 1972/73

[6] D. R. Grigore "*On the Uniqueness of the Non-Abelian Gauge Theories in Epstein-Glaser Approach to Renormalisation Theory*", Romanian J. Phys. **44** (1999) 853-913

[7] D. R. Grigore "*The Standard Model and its Generalisations in Epstein-Glaser Approach to Renormalisation Theory*", Journ. Phys. **A 33** (2000) 8443-8476

[8] D. R. Grigore "*The Standard Model and its Generalisations in Epstein-Glaser Approach to Renormalisation Theory II: the Fermion Sector and the Axial Anomaly*", Journ. Phys **A 34** (2001) 5429-5462

[9] D. R. Grigore, "*Cohomological Aspects of Gauge Invariance in the Causal Approach*", Romanian Journ. Phys. **55** (2010) 386-438

[10] D. R. Grigore, "*Perturbative Gravity in the Causal Approach*", Classical Quant. Gravity **27** (2010) 015013 (33p)

[11] G. Popineau, R. Stora, "*A Pedagogical Remark on the Main Theorem of Perturbative Renormalization Theory*", unpublished preprint

[12] G. Scharf, "*Finite Quantum Electrodynamics: The Causal Approach*", (second edition) Springer, 1995

[13] G. Scharf, "*Quantum Gauge Theories. A True Ghost Story*", John Wiley, 2001 and "*Quantum Gauge Theories - Spin One and Two*", Google books, 2010

[14] R. Stora, "*Lagrangian Field Theory*", Les Houches lectures, Gordon and Breach, N.Y., 1971, C. De Witt, C. Itzykson eds.

[15] O. Steinmann, "*Perturbation Expansions in Axiomatic Field Theory*", Lect. Notes in Phys. **11**, Springer, 1971